Lecture 1 : Introduction and Basic Calculus

Chiara Amorino

19BE02: Calculus and Linear Algebra

Barcelona School of Economics

Course structure

Instructor: Chiara Amorino (chiara.amorino@upf.edu).

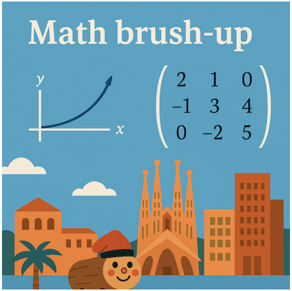
Five lectures: today, Mon, Tue, Wed, Thu.

Each day: 4 hours of lectures, 1 hour of tutorials.

Lectures: 11-1pm and 2-4pm.

Tutorials: 4-5pm.

TA: Yeming Matthew-Ma (yeming.ma@upf.edu).



Grading: This course is not graded. But there will be an exam on Monday, Sept 21st, 10-11am.

Goal of the course

Review core tools of calculus that power modern economics, finance, and data-driven policy.

These tools are crucial in courses on micro/macro, econometrics, finance, and machine learning.

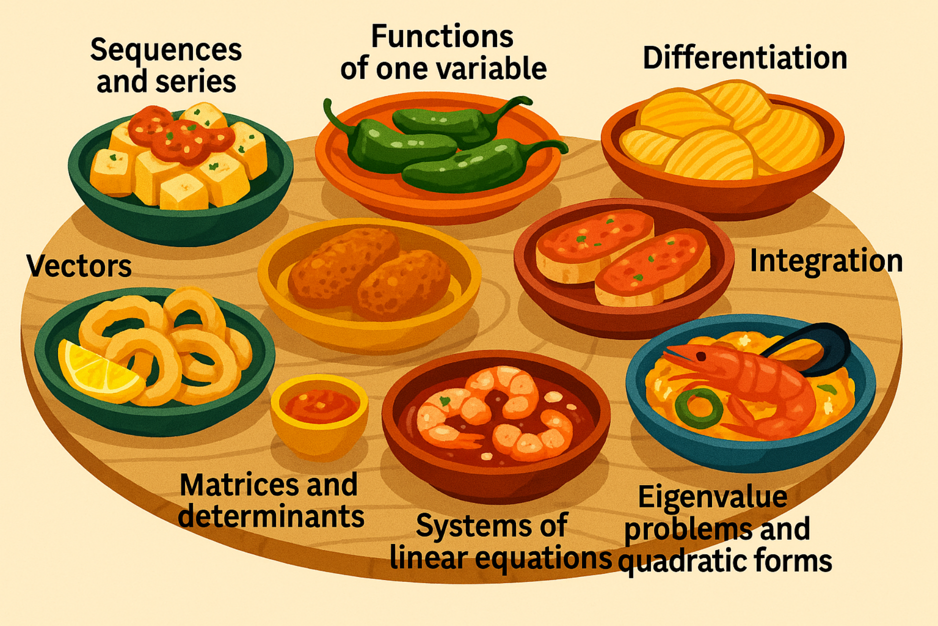
Warm up before the Master starts: sharpen intuition, not just technique.

Main references:

1. *Mathematics for Economists* by C. P. Simon and L. Blume
2. *Mathematics of Economics and Business* by F. Werner and Y. N. Sotskov

(any other standard textbook should be fine)

Course content (math tapas)



Cultivate a Critical Mindset

Technical knowledge requires effort.

- **If a step feels "magical", stop.** Ask *why* it works, not just *how*.
- **Challenge every answer — especially given by LLMs.**
  Large-language models can sound confident yet be wrong; verify with first principles or a textbook.
- **Use multiple lenses.**
  Check special cases, sketch a graph, test numerically, or search for a counterexample.

Sequences

Sequences

A sequence is an ordered list of real numbers $(a_1, a_2, \ldots)$ indexed by $n \in \mathbb{N}$.

Notation: $\{a_n\}_{n=1}^{\infty}$ where each $a_n \in \mathbb{R}$.

Defined by a formula or a recursion.

Examples:

1. $a_n = \dfrac{n-1}{2n+1} \quad (a_1 = 0, a_2 = \frac{1}{5}, a_3 = \frac{2}{7}, \ldots)$
2. $a_1 = 2,\ a_{n+1} = 2a_n - 1 \quad (2, 3, 5, 9, \ldots)$
3. $a_1 = 1, a_2 = 1,\ a_{n+1} = a_n + a_{n-1}$ (Fibonacci)
4. Arithmetic: $a_n = a_1 + (n-1)d$
5. Geometric: $a_n = a_1 r^{n-1}$

Why sequences show up in modern econ/data

Where they appear:

- **User metrics:** daily active users (DAU) over days n.
- **Inflation tracking:** monthly CPI/CPIH index a_n.

Why sequences show up in modern econ/data

Where they appear:

- **User metrics:** daily active users (DAU) over days n.
- **Inflation tracking:** monthly CPI/CPIH index a_n.
- **Training loss:** $a_n =$ loss after n gradient steps (decreasing sequence).
- **Carbon budgets:** remaining allowance after n years.

Read Ch. 2 Werner-Sotskov. Exercises: 2.4, 2.5, 2.7 (Werner-Sotskov)

Small motivating example: Creator economy revenue

Imagine a YouTuber uploading videos one by one. Each video gets a certain number of views:

$$v_1,\ v_2,\ v_3,\ \ldots$$

and YouTube pays the creator a fixed rate of c euros per 1,000 views (RPM).

The revenue from video k is

$$r_k = c \cdot \frac{v_k}{1000}.$$

The total revenue after n videos is then the sequence

$$S_n = r_1 + r_2 + \cdots + r_n.$$

This is also an example of a partial sum; more on that in a second.

Small motivating example: EAR vs APR in FinTech

Let $a_0 = P$ be the initial investment. If a platform advertises an *annual percentage rate* (APR) R but applies interest m times per year, then after n years:

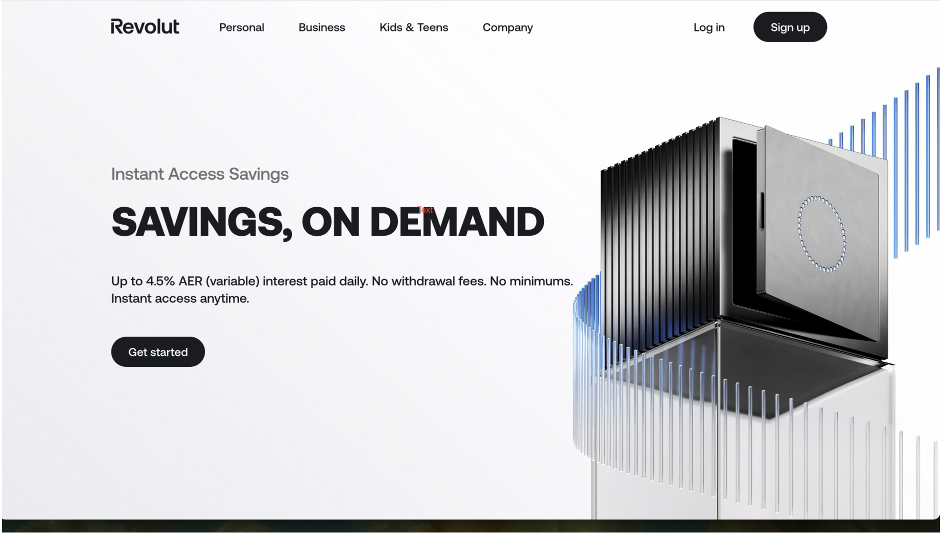
$$a_n = P\left(1+\frac{R}{m}\right)^{mn}.$$

The one-year *effective annual rate* (EAR) is:

$$\frac{a_1 - a_0}{a_0} = \left(1+\frac{R}{m}\right)^{m} - 1.$$

(APR is the quoted yearly rate ignoring compounding; APY/EAR accounts for the fact that interest itself earns interest when applied multiple times in a year.)

What is the actual daily interest? (ignoring fineprint)



Properties of a sequence

A sequence is increasing if $a_{n+1} \geq a_n$ (strictly if $>$) for all $n = 1, 2, \ldots$.

Decreasing if $a_{n+1} \leq a_n$ (strictly if $<$).

Monotone if increasing or decreasing.

Bounded if $\exists C > 0$ s.t. $|a_n| \leq C$ for all n.

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1. $a_n = 2(n-1)^2 - n$ is strictly increasing (since $a_{n+1} - a_n = 4n - 3 > 0$)
2. $a_n = 1/n^2$ is bounded

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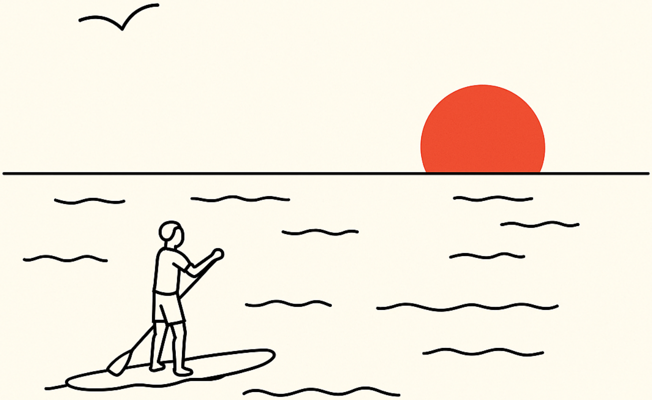
1. $a_n = 2(n-1)^2 - n$ is strictly increasing (since $a_{n+1} - a_n = 4n - 3 > 0$)
2. $a_n = 1/n^2$ is bounded ($0 < a_n \leq 1$)
3. **Training loss:** if a learning rate is sensible, $a_n =$loss often decreases and is bounded below by $0 \Rightarrow$ convergent in many practical cases.

Can a strictly increasing sequence be bounded?



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(equivalently $d_n = |a_n - a|$ converges to zero)

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4. $a_n = r^n \to 0$ if $|r| < 1$; diverges if $r > 1$; no limit if $r \leq -1$.

Formal definition

How do you formalize the statement "gets arbitrarly close"?

Definition: $a = \lim_{n\to\infty} a_n$

$$\forall\epsilon > 0 \quad \exists N_\epsilon \quad \text{s.t. } \forall n > N_\epsilon : \qquad |a_n - a| < \epsilon.$$

Example: $a_n = \dfrac{1}{n} \to 0$. For any $\epsilon > 0$, take $N_\epsilon > \frac{1}{\epsilon}$.

Checks:

- $\lim a_n = a$ if and only if $\lim |a_n - a| = 0$.
- If $a_n = a$ for all n, then $\lim a_n = a$.

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Theorem(Squeeze theorem): Consider three sequences (a_n), (b_n), (c_n).

If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

e.g. $0 < \frac{1}{2n^2-1} \leq \frac{1}{n}$ for $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{2n^2-1} = 0$.

Squeeze theorem

Some limits that matter in practice

Classics

1. $\left(1+\frac{1}{n}\right)^n \to e$
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Example: $\text{APR} = 8\% \quad \Rightarrow \quad \text{EAR} \approx 8.329\%.$

With monthly compounding ($m = 12$) EAR is 8.3%.
With daily compounding ($m = 365$) EAR is 8.328%

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Theorem

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(Just bounded or just monotone is not enough in general)

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Algebra of limits

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n + b_n) = a + b, \quad \lim(a_n b_n) = ab. \qquad \text{Moreover, } \lim \frac{a_n}{b_n} = \frac{a}{b} \text{ if } b_n, b \neq 0.$$

Series

Partial sums and series

The nth partial sum s_n of $\{a_n\}$ is

$$s_n = \sum_{k=1}^{n} a_k.$$

Examples

1. Arithmetic sequence $a_n = a_1 + (n-1)d$: $\quad s_n = na_1 + \frac{n(n-1)}{2}d$

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Use-cases:

- Let *margin* be the profit earned per customer per period and p be the customer retention rate. The total profit from a single customer over n periods is: $s_n = \sum_{k=1}^{n} \text{margin} \cdot p^k$.

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- Let E_k be the emissions produced in period k (e.g., in tonnes of CO_2). The total emissions over n periods are: $s_n = \sum_{k=1}^{n} E_k$, which can be compared to a fixed carbon budget.

Series and convergence

Partial sums also form a sequence. A series is the limit (if it exists) of partial sums:

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Customer Lifetime Values: Consider the example on the previous slide. With margin m and monthly retention $p < 1$, discounted at monthly rate d,

$$\text{CLV} \;=\; \sum_{k=1}^{\infty} \frac{m\,p^{\,k}}{(1+d)^k} \;=\; \frac{m\,\frac{p}{1+d}}{1-\frac{p}{1+d}} \;=\; \frac{mp}{1+d-p}.$$

Chapter 2: Functions of one variable

A one-variable function assigns a unique $y \in \mathbb{R}$ to each $x \in \mathbb{R}$: $\quad y = f(x)$.

x: independent (exogenous) variable; y: dependent (endogenous) variable.

Examples in econ/data: cost, revenue, demand, utility, probability of purchase vs. price, click-through rate vs. ad spend, etc.

Read Ch. 3 Werner-Sotskov; Simon-Blume 2.1-2.2, 5.1-5.3.

Graph and domain

Definition

The graph of f is $\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$.

The domain D is the set of x where $f(x)$ is defined; $D \subseteq \mathbb{R}$.

Notation: $f : D \to \mathbb{R}$.

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Example: $f(x) = \dfrac{1}{x}$ has $D = \mathbb{R} \setminus \{0\}$; $f(x) = \sin(x)$, $D = \mathbb{R}$.

Sometimes we restrict the domain (e.g., prices $x \geq 0$).

Properties of functions

A function $f : D \to \mathbb{R}$ is increasing if $f(x_1) \leq f(x_2)$ for any $x_1 < x_2$ in D (strictly if $f(x_1) < f(x_2)$).

Decreasing: reverse inequality.

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Bounded: $\exists C > 0$ s.t. $|f(x)| \leq C$ for all $x \in D$.

Examples:

- **Price-demand curve** often strictly decreasing on realistic intervals (then zero).
- **Learning curve** typically decreasing as algorithm progresses.

Convex/concave functions

For $x_1 < x_2$ and $t \in [0, 1]$, the convex combination is $tx_1 + (1 - t)x_2$.

A set $D \subset \mathbb{R}$ is convex if it contains convex combinations.

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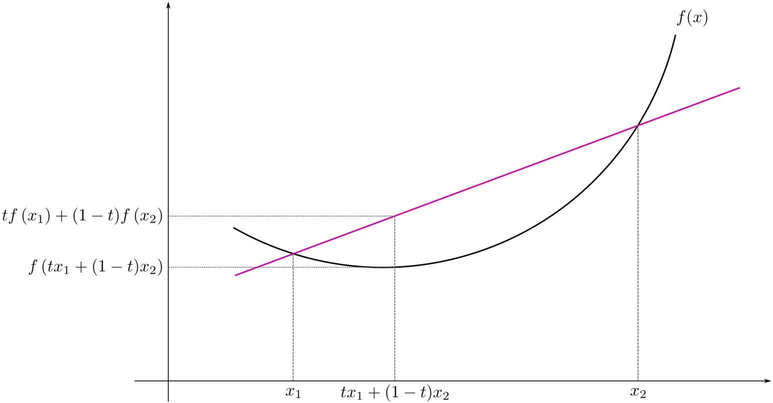
A function $f : D \to \mathbb{R}$ on convex D is convex if

$$f(tx_1 + (1-t)x_2) \quad \leq \quad tf(x_1) + (1-t)f(x_2)$$

(and concave if $\geq$).

For the strict version replace $\leq$ by $<$ for $t \in (0,1)$ and $x_1 \neq x_2$.

Convex function



Quasi-convex/concave functions

f is quasi-convex if

$$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\}.$$

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Facts: convex $\Rightarrow$ quasi-convex; concave $\Rightarrow$ quasi-concave.

Any strictly increasing function is both quasi-convex and quasi-concave.

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Economic examples:

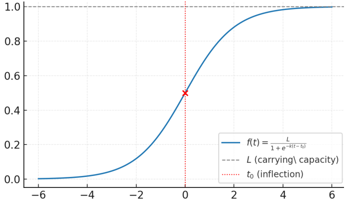
- **Indifference curves:** If a consumer is indifferent between two bundles, any convex combination is at least as good as one of them. This gives quasi-convex preferences.
- **Profits in price:** As a function of price, profits often rise, then fall (too cheap or too expensive reduces profit). This bell-shaped behavior is quasi-concave.

Economic example: logistic growth (adoption curves)

The logistic function with $L > 0$, $k > 0$, $t_0 \in \mathbb{R}$:

$$f(t) = \frac{L}{1 + e^{-k(t-t_0)}}$$

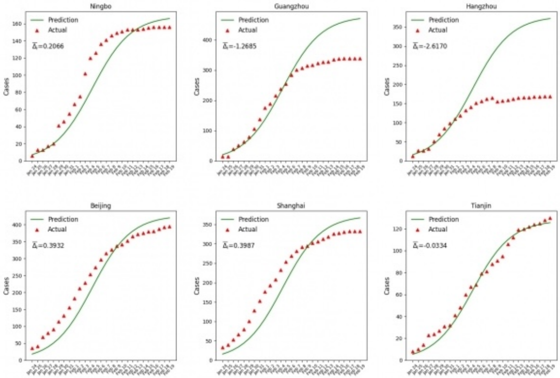
models technology/product adoption or viral spread. Early exponential growth, mid-phase inflection at t_0, saturation near L (market size).

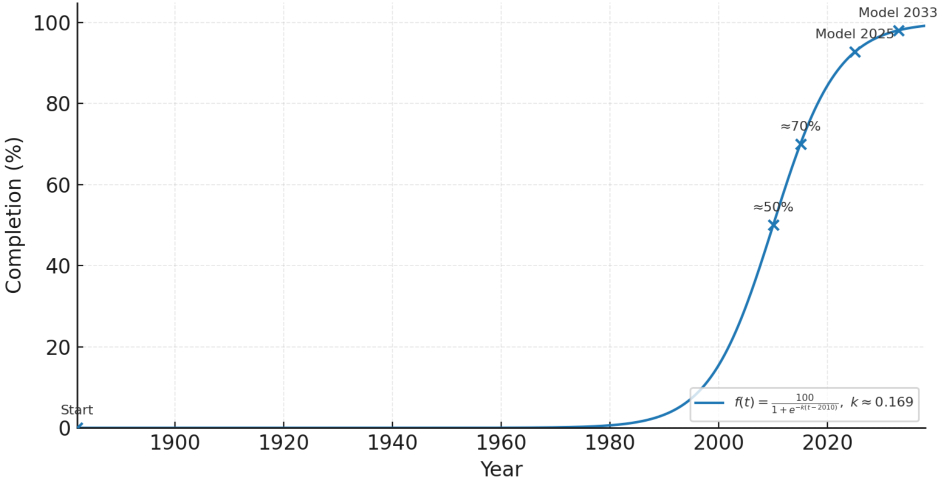


Interpretation: growth accelerates, then slows as saturation/frictions (capacity, regulation, budgets) bind. Same shape often appears in learning curves (accuracy vs. data).

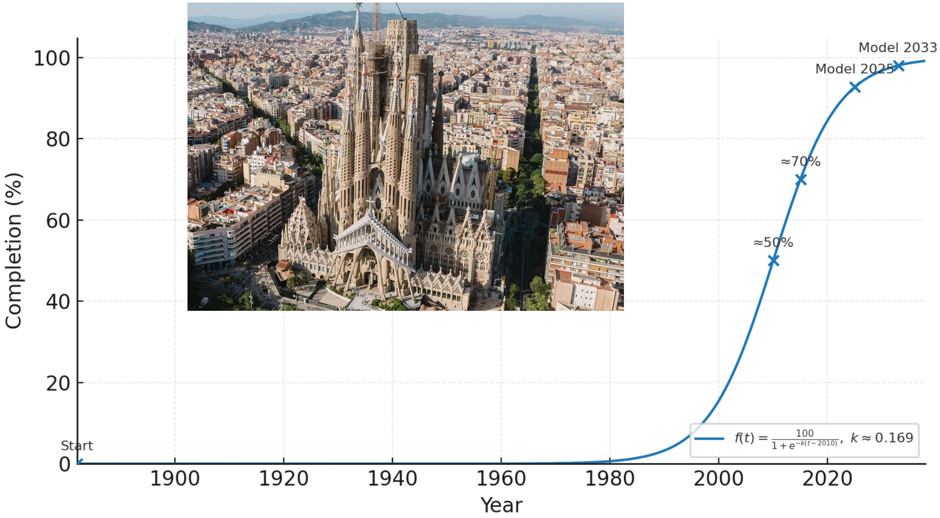
Epidemics and beyond

Logistic curves fit epidemic growth, but also market penetration, platform adoption, content diffusion. Check the following epidemic curves over time of the COVID-19 in China





Model 2033
Model 2025
≈70%
≈50%
Start
$f(t) = \frac{100}{1+e^{-k(t-2010)}},\ k \approx 0.169$
100
80
60
40
20
0
Completion (%)
1900
1920
1940
1960
1980
2000
2020
Year


Completion (%)
Year
0
20
40
60
80
100
1900
1920
1940
1960
1980
2000
2020
Start
≈50%
≈70%
Model 2025
Model 2033
$f(t) = \frac{100}{1+e^{-k(t-2010)}}$, $k \approx 0.169$

Examples of functions

1. Monomials: $f(x) = ax^k, \quad a \in \mathbb{R}$.
2. Polynomials: finite sums of monomials.
3. Rational: ratios of polynomials.
4. Exponential: $f(x) = a^x, \quad a > 0$.
5. Trigonometric: $\sin x$, $\cos x$, ...
6. Linear: $f(x) = ax + b$ (slope a, intercept b; also $y - f(x_0) = a(x - x_0)$).
7. Natural exponential: $f(x) = e^x$ ($e^x = \lim_n (1 + \frac{x}{n})^n$).
8. Natural log: $f(x) = \log x$ ($e^{\log x} = x$, $\log(e^x) = x$)

Examples of functions

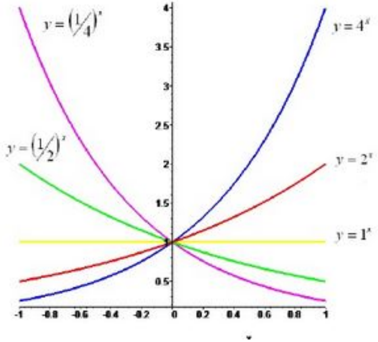
1. Monomials: $f(x) = ax^k, \quad a \in \mathbb{R}$.
2. Polynomials: finite sums of monomials.
3. Rational: ratios of polynomials.
4. Exponential: $f(x) = a^x, \quad a > 0$.
5. Trigonometric: $\sin x$, $\cos x$, $\ldots$
6. Linear: $f(x) = ax + b$ (slope a, intercept b; also $y - f(x_0) = a(x - x_0)$).
7. Natural exponential: $f(x) = e^x$ ($e^x = \lim_n (1 + \frac{x}{n})^n$).
8. Natural log: $f(x) = \log x$ ($e^{\log x} = x$, $\log(e^x) = x$)

Economic examples (often based on the above fundamental examples):

- **Demand curves**: typically decreasing in price (law of demand).
- **Cost curves**: often convex, since producing extra units gets more expensive at higher scale.
- **Utility functions**: often concave, to capture *diminishing marginal utility* (each extra unit of a good adds less satisfaction).
- **Log-sum-exp**: arises in discrete choice models; connects probabilities to utilities.

Exponential functions

$f(x)=1^x$	Horizontal line with y-intercept 1
$f(x)=a^x, \quad 0<a<1$	Exponential decay
$f(x)=a^x, \quad a>1$	Exponential growth



Chapter 3: Differentiation

Differentiation measures sensitivity: how $f(x)$ changes with small changes in x

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x \quad \text{or, equivalently,} \quad \frac{f(x + \Delta x) - f(x)}{\Delta x} \approx f'(x).$$

Examples: price $\rightarrow$ demand sensitivity, ad spend $\rightarrow$ clicks, carbon tax $\rightarrow$ emissions, risk $\rightarrow$ portfolio value.

Read Ch. 4 Werner-Sotskov; Simon-Blume Chs. 2-5.

Exercises: 4.6(c), 4.10, 4.14, 4.15(a)-(c), 4.16(d), 4.17(b) (Werner-Sotskov), 5.5 (b)-(c)-(e) (Simon-Blume)

Functional Limits and Continuity

Limit of a function

Let $f : D \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. We write $L = \lim_{x \to x_0} f(x)$ if

$$x_n \to x_0 \quad \Longrightarrow \quad y_n := f(x_n) \to L.$$

Remark: x_0 need not be in D (removable/essential discontinuities matter in econ too).

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Algebra of limits

If $\lim_{x \to x_0} f(x) = y_1$ and $\lim_{x \to x_0} g(x) = y_2$, then

$$\lim_{x \to x_0} (f(x) + g(x)) = y_1 + y_2, \quad \lim_{x \to x_0} (f(x)g(x)) = y_1 y_2, \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{y_1}{y_2} \text{ if } y_2 \neq 0.$$

Continuity

Definition

f is continuous at $x_0 \in D$ if $\lim_{x \to x_0} f(x) = f(x_0)$. If f is continuous at every point of $B \subset D$, we say f is continuous on B.

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Heine-Borel fact (1D): continuous $f : [a, b] \to \mathbb{R}$ is bounded.

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Non-example

Consider the function $f(x) = \frac{|x|}{x}$ for $x \neq 0$ and $f(0) = 1$. This function is not continuous. Why?

Composition and inverse

Two important results

(composition) If f, g are continuous, so is $h(x) = g(f(x))$ (we also write $h = g \circ f$); moreover

$$\lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x)\right).$$

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(inverse) If $f : [a, b] \to \mathbb{R}$ is strictly monotone and continuous with $f(a) = c, f(b) = d$, then the inverse $f^{-1} : [c, d] \to [a, b]$ is also strictly monotone and continuous.

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Economic example: $p =$ price of a good, $q =$ quantity sold. The *demand function* $q(p)$ gives the quantity q consumers will buy at price p. Equivalently, the *inverse demand function* $p(q)$ gives the price at which q units can be sold. Revenue as a function of quantity is:

$$R(q) = p(q) \cdot q.$$

Differentiation

Derivative

Let $f : (a, b) \to \mathbb{R}$. f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. Denote this limit by $f'(x)$.

Fact: differentiable $\Rightarrow$ continuous.

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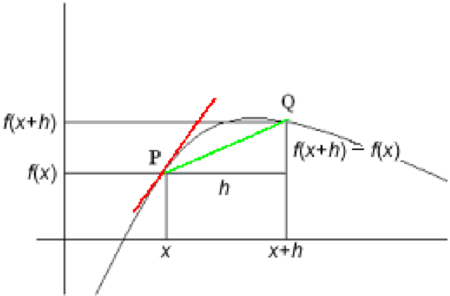
Fact: differentiable $\Rightarrow$ continuous.

The converse does not hold: $|x|$ is continuous but not differentiable at 0 (why?).

Geometric interpretation

If f is differentiable at x_0, the tangent line to the graph of f at $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$



slope of secant through P,Q: $\frac{f(x+h)-f(x)}{h}$; slope of tangent at P: $f'(x)$

A "marginal" is a derivative

In economics, a **marginal** quantity measures how much something changes when we increase an input by a small amount. Mathematically, we often approximate

$$f(x+1) - f(x) \;\approx\; f'(x),$$

when the change is 1 unit.

Examples:

- **Marginal revenue:** $\frac{d}{dq}[p(q)\,q]$ at the current sales level q. The derivative tells you how much extra revenue you gain (or lose) by selling one more unit.
- **Click-through sensitivity:** $\frac{d\,\mathrm{CTR}}{d(\text{ad spend})}$ at the budget currently in use. Here CTR (click-through rate) is a function of ad spending; the derivative measures how much CTR improves with an extra euro of spending.

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If the change is Δx rather than 1: $f(x+\Delta x) - f(x) \approx f'(x)\,\Delta x.$

In differential notation: $df \approx f'(x)\,dx$

Derivatives of elementary functions

1. $c' = 0$
2. $(x^n)' = nx^{n-1}$ for $n \in \mathbb{N}$
3. $(x^\alpha)' = \alpha x^{\alpha-1}$ for $\alpha \in \mathbb{R}$, $x > 0$
4. $(\log x)' = \dfrac{1}{x}$, $x > 0$
5. $(\sin x)' = \cos x$
6. $(\cos x)' = -\sin x$
7. $(e^x)' = e^x$
8. $(a^x)' = a^x \log a$, $a > 0$

Proof of 8: $(a^x)' = (e^{x \log a})' = a^x \log a$ (chain rule)

Product/quotient rules

If f, g are differentiable:

- $(f+g)' = f' + g'$
- $(fg)' = f'g + g'f$ [Leibniz rule]
- $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ where $g \neq 0$ near the point.

Example (revenue): $R(q) = p(q)\,q$, then $R'(q) = p'(q)\,q + p(q)$.

Chain rule

If $h = g \circ f$ and f, g are differentiable at the relevant points, then

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Example (cost of production):

$$C(x) = 4 + \log(x+1) + \sqrt{3x+1} \quad \Rightarrow \quad C'(x) = \frac{1}{x+1} + \frac{3}{2\sqrt{3x+1}}.$$

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Example (choice models): In discrete choice models, the logistic function $\sigma(x) = \frac{1}{1+e^{-x}}$ maps a score x (reflecting how attractive an option is) to the probability of choosing it. Its derivative $\sigma'(x) = \sigma(x)[1 - \sigma(x)]$ measures how sensitive the choice probability is to changes in x.

Differentiating $u(x)^{v(x)}$

Let $u(x) > 0$. For $h(x) = u(x)^{v(x)}$, taking logs gives:

$$\log h = v \log u \quad \Rightarrow \quad \frac{h'}{h} = v' \log u + v\, \frac{u'}{u}.$$

Hence

$$\boxed{h'(x) = u(x)^{v(x)} \Big(v'(x) \log u(x) + v(x)\, \frac{u'(x)}{u(x)} \Big)}$$

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Econ example: A startup's valuation is $V(t) = V_0[1 + g(t)]^{T(t)}$, where $g(t)$ is the projected annual growth rate and $T(t)$ the number of years that growth lasts. Both $g(t)$ and $T(t)$ vary with t.

Some of the contributors (in form of *caganers*)

The inverse function theorem (1D version)

Theorem:

If f is strictly monotone, continuous, and differentiable at x with $f'(x) \neq 0$, then f^{-1} is differentiable at $y = f(x)$ and

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

Since $f^{-1}(f(x)) = x$, chain rule gives $(f^{-1})'(f(x))f'(x) = 1$.

Example: $f(x) = e^x \Rightarrow (\log)'(y) = 1/y$.

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Higher-order derivatives: If f' is differentiable, its derivative is called the second derivative of f and denoted f''. Similarly, we can define higher-order derivatives, and $f^{(n)}$ is called the nth derivative of f.

If $f^{(n)}$ is continuous, we say that f is n times continuously differentiable or C^n for short.

L'Hôpital's rule

Theorem: Let f, g be C^1 on (a, b) with $g' \neq 0$ on (a, b). If

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad \text{or} \quad \pm\infty,$$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \text{(if the RHS limit exists).}$$

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Example:

$$\begin{aligned}
\lim_{x \to 0} (1 + x)^{1/x} &= \lim_{x \to 0} \exp\left(\frac{1}{x} \log(1 + x)\right) \\
&= \exp\left(\lim_{x \to 0} \frac{\log(1 + x)}{x}\right) \\
&= \exp\left(\lim_{x \to 0} \frac{1}{1 + x}\right) = e.
\end{aligned}$$

Optimization in 1D

Monotonicity via derivatives

If f is differentiable on (a, b):

1. f increasing on $[a, b] \iff f'(x) \geq 0$ on (a, b).
2. f decreasing on $[a, b] \iff f'(x) \leq 0$ on (a, b).
3. f constant on $[a, b] \iff f'(x) = 0$ on (a, b).
4. $f'(x) > 0$ on $(a, b) \Rightarrow f$ strictly increasing on $[a, b]$.
5. $f'(x) < 0$ on $(a, b) \Rightarrow f$ strictly decreasing on $[a, b]$.

Converse of 4-5 can fail at isolated points (e.g., x^3).

Example: $f(x) = \frac{x^3}{3} + 2x^2 + 3x + 1$; $f'(x) = (x + 1)(x + 3)$ determines monotonicity intervals and local extrema.

Optimal points: first-order conditions

Local optima

Let $f : D \to \mathbb{R}$. A point $x_0 \in D$ is a local maximum if $\exists (a, b) \subset D$ with $x_0 \in (a, b)$ and $f(x) \leq f(x_0)$ for all $x \in (a, b)$. Similarly for local min with $\geq$.

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If f has a local optimum at interior x and is differentiable, then $f'(x) = 0$.

To see why this is true, stare at $f(x_0 + h) - f(x_0) \approx f'(x_0)h$.

This condition is necessary, not sufficient; see e.g. x^3.

If x is stationary ($f'(x) = 0$) and f is strictly increasing to the left and strictly decreasing to the right, then x is a local max (reverse for min).